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A
NEW METHOD
OF FINDING
FLUENTS BY CONTINUATION.

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A

A NEW METHOD, &c.

ART. I. Put $\dot{F} = \frac{x^m\dot{x}}{a^n+x^n} = x^{m-n}\dot{x} - a^nx^{m-2n}\dot{x} + a^{2n}x^{m-3n}\dot{x} -$ &c. $\pm \frac{a^{vn}x^{m-vn}\dot{x}}{a^n+x^n}$, then $F = \frac{x^{m-n+1}}{m-n+1} - \frac{a^nx^{m-2n+1}}{m-2n+1} + \frac{a^{2n}x^{m-3n+1}}{m-3n+1} -$ &c. $\pm W$, where W repreſents the fluent of the laſt term. Now $\frac{x^m\dot{x}}{\overline{a^n+x^n}^r} = \frac{x^m\dot{x}\times\overline{a^n+x^n}^{1-r}}{a^n+x^n}$; hence $\int\frac{x^m\dot{x}}{\overline{a^n+x^n}^r} = \int\frac{x^m\dot{x}\times\overline{a^n+x^n}^{1-r}}{a^n+x^n} = F \times \overline{a^n+x^n}^{1-r} - \int F \times \frac{\overline{1-r}.nx^{n-1}\dot{x}}{\overline{a^n+x^n}^r} =$ (by ſubſtituting for F its value in the latter quantity) $F \times \overline{a^n+x^n}^{1-r} - \frac{\overline{1-r}.n}{m-n+1} \times \int\frac{x^m\dot{x}}{\overline{a^n+x^n}^r} + \frac{\overline{1-r}.na^n}{m-2n+1} \times \int\frac{x^{m-n}\dot{x}}{\overline{a^n+x^n}^r} - \frac{\overline{1-r}.na^{2n}}{m-3n+1} \times \int\frac{x^{m-2n}\dot{x}}{\overline{a^n+x^n}^r} +$ &c. $\pm \int W \times \frac{\overline{1-r}.nx^{n-1}\dot{x}}{\overline{a^n+x^n}^r}$; transpoſe $-\frac{\overline{1-r}.n}{m-n+1} \times \int\frac{x^m\dot{x}}{\overline{a^n+x^n}^r}$ and divide both ſides of the equation by $\frac{m-rn+1}{m-n+1}$ and we have $\int\frac{x^m\dot{x}}{\overline{a^n+x^n}^r} = \frac{m-n+1}{m-rn+1} \times F \times \overline{a^n+x^n}^{1-r} + \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r}.na^n}{m-2n+1} \times \int\frac{x^{m-n}\dot{x}}{\overline{a^n+x^n}^r} - \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r}.na^{2n}}{m-3n+1} \times \int\frac{x^{m-2n}\dot{x}}{\overline{a^n+x^n}^r} +$ &c. $\pm \frac{m-n+1}{m-rn+1} \times$

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$\int W \times \frac{\overline{1-r}\,.\,nx^{n-1}\dot{x}}{\overline{a^n+x^n}|^{r}}$; now the fluent of the laſt term is $\mp\frac{m-n+1}{m-rn+1} \times W \times \overline{a^n+x^n}|^{1-r} \pm \frac{m-n+1}{m-rn+1} \times \int \frac{a^{vn}x^{m-vn}\dot{x}}{\overline{a^n+x^n}|^{r}}$; hence by ſubſtituting this quantity for the laſt term, it is manifeſt, that the firſt part $\mp \frac{m-n+1}{m-rn+1} \times W \times \overline{a^n+x^n}|^{1-r}$ will be deſtroyed by the laſt term of $\frac{m-n+1}{m-rn+1} \times F \times \overline{a^n+x^n}|^{1-r}$, when we ſubſtitute for F its value; hence if we put $M = \frac{x^{m-n+1}}{m-n+1} - \frac{a^n x^{m-2n-1}}{m-2n+1} + \frac{a^{2n}x^{m-3n+1}}{m-3n+1}$ – &c. omitting the laſt term ± W, we have $\int \frac{x^m\dot{x}}{\overline{x^n+x^n}|^{r}} = \frac{m-n+1}{m-rn+1} \times M \times \overline{a^n+x^n}|^{1-r} + \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r}\,.\,na^n}{m-2n+1} \times \int \frac{x^{m-n}\dot{x}}{\overline{a^n+x^n}|^{r}} - \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r}\,.\,na^{2n}}{m-3n+1} \times \int \frac{x^{m-2n}\dot{x}}{\overline{a^n+x^n}|^{r}} + \&c. \pm \frac{m-n+1}{m-rn+1} \times a^{vn} \times \int \frac{x^{m-vn}\dot{x}}{\overline{a^n+x^n}|^{r}}$; hence, if the fluent of the laſt term be given, we have the general law of continuation by which we may find the fluent of $\frac{x^m\dot{x}}{\overline{a^n+x^n}|^{r}}$. If the fluxion be $\frac{x^m\dot{x}}{\overline{a^n+x^n}|^{r}}$ all the terms after the firſt will be negative, and the laſt always poſitive.

Ex. 1. Given the fluent of $\frac{\dot{x}}{\sqrt{1+x^2}}$ to find the fluent of $\frac{x^{2s}\dot{x}}{\sqrt{1+x^2}}$.

Here $n=2$, $a=1$, $m=2s$, $r=\frac{1}{2}$, $M = \frac{x^{2n-1}}{2n-1} - \frac{x^{2n-3}}{2n-3}$ + &c. to $\pm x$,

$\pm a$, and the fluent of $\frac{\dot{x}}{\sqrt{1+x^2}}$ is the hyp. log. $\overline{x+\sqrt{1+x^2}}$ which call Q; hence $\int\frac{x^{2s}\dot{x}}{\sqrt{1+x^2}}=\frac{2s-1}{2s}\times M\times\sqrt{1+x^2}+\frac{2s-1}{2s\cdot 2s-3}\times\int\frac{x^{2s-2}\dot{x}}{\sqrt{1+x^2}}\pm\frac{2s-1}{2s\cdot 2s-5}\times\int\frac{x^{2s-4}\dot{x}}{\sqrt{1+x^2}}+\text{\&c.}\pm\frac{2s-1}{2s}\times Q.$

If $s=1$, $\int\frac{x^2\dot{x}}{\sqrt{1+x^2}}=\frac{1}{2}\sqrt{1+x^2}\times x-\frac{1}{2}Q=\alpha.$

$s=2$, $\int\frac{x^4\dot{x}}{\sqrt{1+x^2}}=\frac{3}{4}\sqrt{1+x^2}\times\overline{\frac{x^3}{3}-x}+\frac{3}{4}\alpha+\frac{3}{4}Q=\beta.$

$s=3$, $\int\frac{x^6\dot{x}}{\sqrt{1+x^2}}=\frac{5}{6}\sqrt{1+x^2}\times\overline{\frac{x^5}{5}-\frac{x^3}{3}+x}+\frac{5}{6\cdot 3}\beta-\frac{5}{6}\alpha-\frac{5}{6}Q=\gamma.$

$s=4$, $\int\frac{x^8\dot{x}}{\sqrt{1+x^2}}=\frac{7}{8}\sqrt{1+x^2}\times\overline{\frac{x^7}{7}-\frac{x^5}{5}+\frac{x^3}{3}-x}+\frac{7}{8\cdot 5}\gamma-\frac{7}{8\cdot 3}\beta+\frac{7}{8}\alpha+\frac{7}{8}Q.$

&c. &c. &c.

Ex. 2. To find the fluent of $x^{\frac{s}{2}}\dot{x}\sqrt{2+x}$, given the fluent of $x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}$, and s an odd number.

Here $a=2$, $n=1$, $r=-\frac{1}{2}$, $\frac{s}{2}=m$, $m-vn=-\frac{1}{2}$, or $\frac{s}{2}-v=-\frac{1}{2}$; $\therefore v=\frac{s+1}{2}$, $M=\frac{2x^{\frac{s}{2}}}{s}-\frac{4x^{\frac{s}{2}-1}}{s-2}+\frac{8x^{\frac{s}{2}-2}}{s-4}-$ &c. and the fluent (Q) of $x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}$ is $\pi+\sqrt{2x+x^2}$, where $\pi=$ hyp. log. $\overline{1+x+\sqrt{2x+x^2}}$; hence $\int x^{\frac{s}{2}}\dot{x}\sqrt{2+x}=\frac{s}{s+3}\times\overline{2+x}|^{\frac{3}{2}}\times M+\frac{s}{s+3}\times\frac{b}{s\;\;2}\times\int x^{\frac{s}{2}-1}\dot{x}\sqrt{2+x}-\frac{s}{s+3}\times\frac{12}{s-4}\times\int x^{\frac{s}{2}-2}\dot{x}\sqrt{2+x}+\frac{s}{s+3}\times\frac{24}{s-6}\times\int x^{\frac{s}{2}-3}\dot{x}\sqrt{2+x}-\text{\&c.}\pm\frac{s}{s+3}\times 2^{\frac{s+1}{2}}\times\int x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}.$

If

If $s = 1, \int x^{\frac{1}{2}}\dot{x}\sqrt{2+x} = \frac{1}{4} \times \overline{2+x}|^{\frac{3}{2}} \times 2x^{\frac{1}{2}} - \frac{1}{2}Q = \alpha.$

$s = 3, \int x^{\frac{3}{2}}\dot{x}\sqrt{2+x} = \frac{1}{2} \times \overline{2+x}|^{\frac{3}{2}} \times \overline{\frac{2x^{\frac{3}{2}}}{3} - 4x^{\frac{1}{2}}} + 3\alpha + 2Q = \beta.$

$s = 5, \int x^{\frac{5}{2}}\dot{x}\sqrt{2+x} = \frac{5}{8} \times \overline{2+x}|^{\frac{3}{2}} \times \overline{\frac{2x^{\frac{5}{2}}}{5} - \frac{4x^{\frac{3}{2}}}{3} + 8x^{\frac{1}{2}}} + \frac{5}{4}\beta -$

$\frac{15}{2}\alpha - 5Q = \gamma.$

$s = 7, \int x^{\frac{7}{2}}\dot{x}\sqrt{2+x} = \frac{7}{10} \times \overline{2+x}|^{\frac{3}{2}} \times \overline{\frac{2x^{\frac{7}{2}}}{7} - \frac{4x^{\frac{5}{2}}}{5} + \frac{8x^{\frac{3}{2}}}{3} - 16x^{\frac{1}{2}}} +$

$\frac{21}{25}\gamma - \frac{14}{5}\beta + \frac{84}{5}\alpha + \frac{56}{5}Q.$

&c. &c. &c.

II. Let $\dot{F}\,\frac{x^n\dot{x}}{a+bx^m+x^{2m}} = x^{n-2m}\dot{x} - Px^{n-3m}\dot{x} + Qx^{n-4m}\dot{x} -$ &c. $\pm$ $\frac{Vx^{n-sm}\dot{x}}{a+bx^m+x^{2m}} \pm \frac{Wx^{n-\overline{s+1}\,.\,m}\dot{x}}{a+bx^m+x^{2m}}$, then $F = \frac{x^{n-2m+1}}{n-2m+1} - \frac{Px^{n-3m-1}}{n-3m+1} + \frac{Qx^{n-4m+1}}{n-4m+1}$ $-$ &c. $\pm T \pm U$, where T and U are put for the fluents of the two laſt terms, and P, Q, &c. for the co-efficients ariſing from the diviſion. Now, $\int \frac{x^n\dot{x}}{\sqrt{a+bx^m+x^{2m}}} = \int \frac{x^n\dot{x} \times \overline{a+bx^m+x^{2m}}|^{1-r}}{a+bx^m+x^{2m}} =$ $F \times \overline{a+bx^m+x^{2m}}|^{1-r} - \int F + \frac{\overline{1-r}\,.\,mbx^{m-1}\dot{x} + \overline{1-r}\,.\,2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} =$ (by ſubſtituting for F its value in the latter quantity, and putting A, B, C, &c for the co-efficients which ariſe in conſequence thereof) $F \times \overline{a+bx^m+x^{2m}}|^{1-r} - A \times \int \frac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} + B \times$ $\int \frac{x^{n-m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} - C \times \int \frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} +$ &c.

$$\mp\int \mathrm{T}\times\frac{\overline{1-r}\,.\,mbx^{m-1}\dot{x}+\overline{1-r}\,.\,2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}\rceil^{r}}$$

$\mp\int \mathrm{U}\times\frac{\overline{1-r}\,.\,mbx^{m-1}\dot{x}+\overline{1-r}\,.\,2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}\rceil^{r}}$; hence by tranſpoſition and diviſion we have $\int\frac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}\rceil^{r}}=\frac{1}{1+\mathrm{A}}\times\mathrm{F}\times\overline{a+bx^m+x^{2m}}\rceil^{1-r}$ $+\ \frac{\mathrm{B}}{1+\mathrm{A}}\times\int\frac{x^{n-m}\dot{x}}{\overline{a+bx^m+x^{2m}}\rceil^{r}}-\frac{\mathrm{C}}{1+\mathrm{A}}\times\int\frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+x^{2m}}\rceil^{r}}+$ &c.

$$\mp\int\frac{\mathrm{T}}{1+\mathrm{A}}\times\frac{\overline{1-r}\,.\,mbx^{m-1}\dot{x}+\overline{1-r}\,.\,2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}\rceil^{r}}$$

$\mp\int\frac{\mathrm{U}}{1+\mathrm{A}}\times\frac{\overline{1-r}\,.\,mbx^{m-1}\dot{x}+\overline{1-r}\,.\,2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}\rceil^{r}}$. Now the fluents of theſe two laſt terms are $\mp\frac{\mathrm{T}}{1+\mathrm{A}}\times\overline{a+bx^m+x^{2m}}\rceil^{1-r}\pm\frac{\mathrm{V}}{1+\mathrm{A}}$ $\int\frac{x^{n-sm}\dot{x}}{\overline{a+bx^m+x^{2m}}\rceil^{r}}$ and $\mp\frac{\mathrm{U}}{1+\mathrm{A}}\times\overline{a+bx^m+x^{2m}}\rceil^{1-r}\pm\frac{\mathrm{W}}{1+\mathrm{A}}\int\frac{x^{n-\overline{s+1}\,.\,m}\dot{x}}{\overline{a+bx^m+x^{2m}}\rceil^{r}}$ reſpectively; hence, by ſubſtituting theſe for the laſt term, it is manifeſt that $\mp\frac{\mathrm{T}}{1+\mathrm{A}}\times\overline{a+bx^m+x^{2m}}\rceil^{1-r}$ and $\mp\frac{\mathrm{U}}{1+\mathrm{A}}$ $\times\overline{a+bx^m+x^{2m}}\rceil^{1-r}$ will be deſtroyed by the two laſt terms of $\frac{1}{1+\mathrm{A}}\times\mathrm{F}\times\overline{a+bx^m+x^{2m}}\rceil^{1-r}$ when we ſubſtitute for F its value; hence, if we put $\mathrm{M}=\frac{x^{n-2m-1}}{n-2m+1}-\frac{\mathrm{P}x^{n-3m+1}}{n-3m+1}+\frac{\mathrm{Q}x^{n-4m+1}}{n-4m+1}-$ &c. omitting the two laſt terms $\pm$ T and $\pm$ U, we ſhall have $\int\frac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}\rceil^{r}}=\frac{1}{1+\mathrm{A}}\times\mathrm{M}\times\overline{a+bx^m+x^{2m}}\rceil^{1-r}+\frac{\mathrm{B}}{1+\mathrm{A}}\times$ $\int\frac{x^{n-m}\dot{x}}{\overline{a+bx^m+x^{2m}}\rceil^{r}}-\frac{\mathrm{C}}{1+\mathrm{A}}\times\int\frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+x^{2m}}\rceil^{r}}+$ &c. $\pm\frac{\mathrm{V}}{1+\mathrm{A}}\times$

$\int \frac{x^{n-sm}\dot{x}}{\overline{a+bx^m+x^{2m}}|^{r}} \pm \frac{W}{1+A} \times \frac{x^{n-\overline{s+1}.m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^{r}}$; hence if the two laſt fluents be given, we have the general law of continuation up to $\frac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}|^{r}}$ in the ſame manner as before.

III. In general, if we proceed as in the two laſt articles, we ſhall find $\int \frac{x^n\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^{r}} = \frac{M}{P} \times \overline{a+bx^m+\&c.\ x^{tm}}|^{1-r} + \frac{A}{P} \times \int \frac{x^{n-m}\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^{r}} + \frac{B}{P} \times \int \frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^{r}} + \&c. \pm \frac{T}{P} \times \int \frac{x^{n-sm}\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^{r}} \pm \frac{V}{P} \times \int \frac{x^{n-\overline{s+1}.m}\dot{x}}{\overline{a+bx^m+\&c.\ x^{tm}}|^{r}} \pm \&c.$ where the number of theſe laſt terms is t, and $M = \frac{x^{n-tm+1}}{n-tm+1} - \frac{Qx^{n-\overline{t+1}.m-r}}{n-t+1.m+1} + \&c.$ omitting, as before, the terms at the end ariſing from the remainders. Hence if the laſt t fluents be given, we can by continuation find the required fluent.

Becauſe the diviſion of $\frac{x^n\dot{x}}{a+bx^m+\&c.\ x^{tm}}$ may be expreſſed by an *aſcending* ſeries $x^n\dot{x} - Qx^{n+m}\dot{x} + Rx^{n+2m}\dot{x} - \&c.$ it is manifeſt, that by the ſame method we may continue the fluents downwards as well as upwards.

IV. Let $\dot{F} = \frac{x^n\dot{x}}{1-x} = -x^{n-1}\dot{x} - x^{n-2}\dot{x} - x^{n-3}\dot{x} - \&c. - x^{n-r}\dot{x} + \frac{x^{n-r}\dot{x}}{1-x}$, then $F = -\frac{x^n}{n} - \frac{x^{n-1}}{n-1} - \frac{x^{n-2}}{n-2} - \&c. - \frac{x^{n-r+1}}{n-r+1} + W$, where W is the fluent of the laſt term. Now $\frac{x^n\dot{x}}{\sqrt{1-x^2}} = \frac{x^n\dot{x}}{1-x} \times \sqrt{\frac{1-x}{1+x}}$,

hence

hence $\int \frac{x^n\dot{x}}{\sqrt{1-x^2}} = \int \frac{x^n\dot{x}}{1-x} \times \sqrt{\frac{1-x}{1+x}} = \mathrm{F} \times \sqrt{\frac{1-x}{1+x}} + \int \mathrm{F} \times \frac{\dot{x}}{\sqrt{1+x^2}\times\overline{1+x}} = \mathrm{F} \times \sqrt{\frac{1-x}{1+x}} - \int \frac{x^n\dot{x}}{n\sqrt{1-x^2}\times\overline{1+x}} - \int \frac{x^{n-1}\dot{x}}{\overline{n-1}\,.\,\sqrt{1-x^2}\times\overline{1+x}} - \int \frac{x^{n-2}\dot{x}}{\overline{n-2}\,.\,\sqrt{1-x^2}\times\overline{1+x}} - \&c. - \int \frac{x^{n-r+1}\dot{x}}{\overline{n-r+1}\,.\,\sqrt{1-x^2}\times\overline{1+x}} + \int \mathrm{W} \times \frac{\dot{x}}{\sqrt{1-x^2}\times\overline{1+x}}$. But

$$\frac{x^n\dot{x}}{n\sqrt{1-x^2}\times\overline{1+x}} = \frac{x^{n-1}\dot{x}}{n\sqrt{1-x^2}} - \frac{x^{n-2}\dot{x}}{n\sqrt{1-x^2}} + \frac{x^{n-3}\dot{x}}{n\sqrt{1-x^2}} - \&c. \mp \frac{x^{n-r}\dot{x}}{n\sqrt{1-x^2}} \pm \frac{x^{n-r}\dot{x}}{n\sqrt{1-x^2}\times\overline{1+x}}$$

$$\frac{x^{n-1}\dot{x}}{\overline{n-1}\,.\,\sqrt{1-x^2}\times\overline{1+x}} = \frac{x^{n-2}\dot{x}}{\overline{n-1}\,.\,\sqrt{1-x^2}} - \frac{x^{n-3}\dot{x}}{\overline{n-1}\,.\,\sqrt{1-x^2}} + \&c. \pm \frac{x^{n-r}\dot{x}}{\overline{n-1}\,.\,\sqrt{1-x^2}} \mp \frac{x^{n-r}\dot{x}}{\overline{n-1}\,.\,\sqrt{1-x^2}\times\overline{1+x}}$$

$$\frac{x^{n-2}\dot{x}}{\overline{n-2}\,.\,\sqrt{1-x^2}\times\overline{1+x}} = \frac{x^{n-3}\dot{x}}{\overline{n-2}\,.\,\sqrt{1-x^2}} - \&c. \mp \frac{x^{n-r}\dot{x}}{\overline{n-2}\,.\,\sqrt{1-x^2}} \pm \frac{x^{n-r}\dot{x}}{\overline{n-2}\,.\,\sqrt{1-x^2}\times\overline{1+x}}$$

&c. &c. &c.

$$\frac{x^{n-r+1}\dot{x}}{\overline{n-r+1}\,.\,\sqrt{1-x^2}\times\overline{1+x}} = \frac{x^{n-r}\dot{x}}{\overline{n-r+1}\,.\,\sqrt{1+x^2}} - \frac{x^{n-r}\dot{x}}{\overline{n-r+1}\,.\,\sqrt{1-x^2}\times\overline{1+x}}.$$

Hence $\int \frac{x^n\dot{x}}{\sqrt{1-x^2}} = \mathrm{F} \times \sqrt{\frac{1-x}{1+x}} - \frac{1}{n} \times \int \frac{x^{n-1}\dot{x}}{\sqrt{1-x^2}} + \overline{\frac{1}{n} - \frac{1}{n-1}} \times \int \frac{x^{n-2}\dot{x}}{\sqrt{1-x^2}} - \overline{\frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-2}} \times \int \frac{x^{n-3}\dot{x}}{\sqrt{1-x^2}} + \&c. \overline{\pm \frac{1}{n} \mp \frac{1}{n-1} \pm \&c. - \frac{1}{n-r+1}} \times \int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}} \overline{\mp \frac{1}{n} \pm \frac{1}{n-1} \mp \&c. + \frac{1}{n-r+1}} \times \int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}\times\overline{1+x}} + \int \mathrm{W} \times \frac{\dot{x}}{\sqrt{1-x^2}\times\overline{1+x}}$. Now $\int \mathrm{W} \times \frac{\dot{x}}{\sqrt{1-x^2}\times\overline{1+x}} = -\mathrm{W} \times \overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}} + \int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}}$; alſo $\int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}\times\overline{1+x}} = -x^{n-r} \times \sqrt{\frac{1-x}{1+x}} + \overline{n-r} \times \int \frac{x^{n-r-1}\dot{x}}{\sqrt{1-x^2}} - \overline{n-r} \times \int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}}$; hence, by ſubſtituting theſe quantities in the two laſt terms, it is manifeſt that $-\mathrm{W} \times \overline{\frac{1-x}{1-x}}\Big|^{\frac{1}{2}}$ will be deſtroyed by the laſt term of

F ×

$\mathrm{F}\times\overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}}$ when we ſubſtitute for F its value; therefore, if we put $\mathrm{M}=-\frac{x^n}{n}-\frac{x^{n-1}}{n-1}-\frac{x^{n-2}}{n-2}-\&c.-\frac{x^{n-r+1}}{n-r+1}$, we ſhall have

$$\int\frac{x^n\dot{x}}{\sqrt{1\,\text{[illegible]}\,x^2}}=\overline{\mathrm{M}\pm\frac{1}{n}\mp\frac{1}{n-1}\pm\&c.-\frac{1}{n-r+1}}\times\overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}}-\frac{1}{n}\times\int\frac{x^{n-1}\dot{x}}{\sqrt{1-x^2}}$$

$$+\overline{\frac{1}{n}-\frac{1}{n-1}}\times\int\frac{x^{n-2}\dot{x}}{\sqrt{1-x^2}}-\overline{\frac{1}{n}-\frac{1}{n-1}+\frac{1}{n-2}}\times\int\frac{x^{n-3}\dot{x}}{\sqrt{1-x^2}}+\&c.$$

$$\overline{\overline{\pm\frac{1}{n}\mp\frac{1}{n-1}\pm\&c.-\frac{1}{n-r+1}\times\overline{n-r+1}+1}}\times\int\frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}}\mp$$

$$\overline{\frac{1}{n}\pm\frac{1}{n-1}\mp\&c.+\frac{1}{n-r+1}}\times\overline{n-r}\times\int\frac{x^{n-r-1}\dot{x}}{\sqrt{1-x^2}}.$$

Hence, if the two laſt fluents be given, we have the general law of continuation up to the fluent of $\frac{x^n\dot{x}}{\sqrt{1-x^2}}$, where the index of x without the vinculum increaſes by unity each time. And in the ſame manner we may (by increaſing the index of x without by m) find the fluent of $\frac{x^n\dot{x}}{\sqrt{a^{2m}-x^{2m}}}$ if we have given the fluents of $\frac{x^{n-rm}\dot{x}}{\sqrt{a^{2m}-x^{2m}}}$ and $\frac{x^{n-\overline{r+1}\cdot m}\dot{x}}{\sqrt{a^{2m}-x^{2m}}}$. Thus we have a general law of continuation, where the index of x without is increaſed by half the index under the vinculum.

V. Aſſume $\dot{\mathrm{F}}=\frac{x^n\dot{x}}{x^m-b}=x^{n-m}\dot{x}+bx^{n-2m}\dot{x}+b^2x^{n-3m}\dot{x}+\&c.+\frac{b^rx^{n-rm}\dot{x}}{x^m-b}$, then $\mathrm{F}=\frac{x^{n-m+1}}{n-m+1}+\frac{bx^{n-2m+1}}{n-2m+1}+\frac{b^2x^{n-3m+1}}{n+3m+1}+\&c.+\mathrm{W}$, where W is put for the fluent of the laſt term. Now

$$\int x^n\dot{x}\sqrt{\frac{x^m-a}{x^m-b}}=\int\frac{x^n\dot{x}}{x^m-b}\times\sqrt{\overline{x^m-a}\times\overline{x^m-b}}=\mathrm{F}\times\sqrt{\overline{x^m-a}\times\overline{x^m+b}}$$

—

$-\int F\times\frac{2mx^{2m-1}\dot{x}-\overline{a+b},\,mx^{m-1}\dot{x}}{2\sqrt{\overline{x^m-a}\times\overline{x^m-b}}}$ = (by ſubſtituting for F its value in the latter quantity, and putting A, B, C, &c. for the co-efficients which ariſe therefrom) $F\times\sqrt{\overline{x^m-a}\times\overline{x^m-b}}-A\int\frac{x^n\dot{x}}{\sqrt{\overline{x^m-a}\times\overline{x^m-b}}}-B\int\frac{x^{n-m}\dot{x}}{\sqrt{\overline{x^m-a}\times\overline{x^m-b}}}-C\int\frac{x^{n-2m}\dot{x}}{\sqrt{\overline{x^m-a}\times\overline{x^m-b}}}-\&c.-\int W\times\frac{2mx^{2m-1}\dot{x}-\overline{a+b}\,.\,mx^{m-1}\dot{x}}{2\sqrt{\overline{x^m-a}\times\overline{x^m-b}}}$. Now $\frac{x^n\dot{x}}{\sqrt{\overline{x^m-a}\times\overline{x^m-b}}}=\frac{x^n\dot{x}}{x^m-a}\sqrt{\frac{x^m-a}{x^m-b}}$, $\frac{x^{n-m}\dot{x}}{\sqrt{\overline{x^m-a}\times\overline{x^m-b}}}=\frac{x^{n-m}\dot{x}}{x^m-a}\sqrt{\frac{x^m-a}{x^m-b}}$, $\frac{x^{n-2m}\dot{x}}{\sqrt{\overline{x^m-a}\times\overline{x^m-b}}}=\frac{x^{n-2m}\dot{x}}{x^m-a}\sqrt{\frac{x^m-a}{x^m-b}}$, &c. But

$$\begin{array}{lr}
\frac{x^n\dot{x}}{x^m-a}= x^{n-m}\dot{x}+dx^{n-2m}\dot{x} & +a^2x^{n-3m}\dot{x}+\&c.+a^{r-1}x^{n-rm}\dot{x}+\frac{a^rx^{n-rm}\dot{x}}{x^m-a}\\
\frac{x^{n-m}\dot{x}}{x^m-a}= & x^{n-2m}\dot{x}+ax^{n-3m}\dot{x}+\&c.+a^{r-2}x^{n-rm}\dot{x}+\frac{a^{r-1}x^{n-rm}\dot{x}}{x^m-a}\\
\frac{x^{n-2m}\dot{x}}{x^m-a}= & x^{n-3m}\dot{x}+\&c.+a^{r-3}x^{n-rm}\dot{x}+\frac{a^{r-2}x^{n-rm}\dot{x}}{x^m-a}\\
\&c. \qquad\qquad \&c. & \&c.
\end{array}$$

Hence, by ſubſtitution, $\int x^n\dot{x}\sqrt{\frac{x^m-a}{x^m-b}}=F\times\sqrt{\overline{x^m-a}\times\overline{x^m-b}}$ $-A\int x^{n-m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}}-\overline{Aa+B}\times\int x^{n-2m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}}-\overline{Aa^2+Ba+C}$ $\times\int x^{n-3m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}}-\&c.-\overline{Aa^{r-1}+Ba^{r-2}+Ca^{r-3}+\&c.}\times$ $\times\int x^{n-rm}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}}-\overline{Aa^r+Ba^{r-1}+Ca^{r-2}+\&c.}\times\int\frac{x^{n-rm}\dot{x}}{\sqrt{\overline{x^m-a}+\overline{x^m-b}}}$ $-\int W\times\frac{2mx^{2m-1}\dot{x}-\overline{a+b}\,.\,mx^{m-1}\dot{x}}{2\sqrt{\overline{x^m-a}\times\overline{x^m-b}}}$. But

$- \int W \times \frac{2mx^{2m-1}\dot{x} - \overline{a+b}.mx^{m-1}\dot{x}}{2\sqrt{\overline{x^m-a} \times \overline{x^m-b}}}$ is $- W \times \sqrt{\overline{x^m-a} \times \overline{x^m-b}}$ $+ b^r \times \int x^{n-rm}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}}$; hence, by ſubſtituting this for the laſt term, it is manifeſt, that $-W \times \sqrt{\overline{x^m-a} \times \overline{x^m-b}}$ will be deſtroyed by the laſt term of $F \times \sqrt{\overline{x^m-a} \times \overline{x^m-b}}$ when we ſubſtitute for F its value; therefore, if we put $M = \frac{x^{n-m+1}}{n-m+1} + \frac{bx^{n-2m+1}}{n-2m+1} + \&c. + \frac{b^{r-1}x^{n-rm+1}}{n-rm+1}$, we have $\int x^n\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} = M \times \sqrt{\overline{x^m-a} \times \overline{x^m-b}} - A\int x^{n-m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa+B} \times \int x^{n-2m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa^2+Ba+C} \times \int x^{n-3m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \&c. - \overline{Aa^{r-1}+Ba^{r-2}+\&c.-b^r} \times \int x^{n-rm}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa^r+Ba^{r-1}+Ca^{r-2}+\&c.} \times \int \frac{x^{n-rm}\dot{x}}{\sqrt{\overline{x^m-a} \times \overline{x^m-b}}}$. Hence if the laſt two fluents be given, we have the general law of continuation up to the fluent of $x^n\dot{x}\sqrt{\frac{x^m-a}{x^m-b}}$.

The utility of finding fluents by continuation was manifeſt to Sir ISAAC NEWTON, who firſt propoſed it; and ſince his time ſome of the moſt eminent mathematicians have employed much of their attention upon it. The method which I have inveſtigated and exemplified in this Paper I offer as being entirely new; and at the ſame time it not only exhibits, at once, the general law up to the required fluent, but alſo appears, from ſome of the inſtances here given, to be more extenſive and convenient in its application than any method hitherto offered.

The general reſolution of the given fluxion into a ſeries of fluxions of the ſame kind, where the index of the unknown quantity without the vinculum keeps decreaſing or increaſing either by the index under or by half the index, has not, that I know of, before been given; which furniſhes us at once not only with a very eaſy method of *continuing* fluents, but alſo points out a very ſimple method of inveſtigating the fluent of the given fluxion *without* continuation. For if $\int \dot{A} = p + b\int \dot{B} + c\int \dot{C} + d\int \dot{D} + \&c.$ $\int \dot{B} = p' + c'\int \dot{C} + d'\int \dot{D} + \&c.$ $\int \dot{C} = p'' + d''\int \dot{D} + \&c.$ &c. &c. then if for $\int \dot{B}$, $\int \dot{C}$, &c. &c. we ſubſtitute their reſpective values, we ſhall get a general ſeries for $\int \dot{A}$ without continuation. The extent of any new method is, at firſt, ſeldom obvious; and how far that which is here propoſed may be ſucceſsfully employed in other caſes will beſt appear from its application. Different methods will always be found to have their uſes in particular caſes; for where one becomes impracticable another will often be found to ſucceed; and I hope that which is here offered will contribute ſomething towards facilitating the inveſtigation of fluents.